

# Hypohamiltonian planar cubic graphs with girth five

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## Abstract

A graph is called hypohamiltonian if it is not hamiltonian but becomes hamiltonian if any vertex is removed. Many hypohamiltonian planar cubic graphs have been found, starting with constructions of Thomassen in 1981. However, all the examples found until now had 4-cycles. In this note we present the first examples of hypohamiltonian planar cubic graphs with cyclic connectivity five, and thus girth five. We show by computer search that the smallest members of this class are three graphs with 76 vertices.

**Keywords:** graph generation; hypohamiltonian graph; planar graph; cubic graph

**MSC 2010.** 05C10, 05C30, 05C38, 05C45, 05C85.

## 1 Introduction

A graph is called *hypohamiltonian* if it is not hamiltonian but becomes hamiltonian if any vertex is removed. Such graphs exist on orders 10 (the Petersen graph), 13, 15, 16, and all orders from 18 onwards [1]. It is elementary to show that hypohamiltonian graphs must be 3-connected and have cyclic connectivity at least 4.

A substantial body of literature is devoted to finding hypohamiltonian graphs with various properties. The first planar hypohamiltonian graph, with 105 vertices, was found by Thomassen [8]; the present smallest known order is 40 vertices but smaller orders have not been ruled out [7].

Planar hypohamiltonian graphs can be cubic, as first shown by Thomassen with an example on 94 vertices [9]. The smallest examples found so far have 70 vertices; the first example by Araya and Wiener [2] and six more by Jooyandeh and McKay [6].

Planar hypohamiltonian graphs can also have girth 5, and in this case the smallest order is proven to be 45 [7]. However, it was not known until now whether planar hypohamiltonian graphs can be both cubic and of girth 5. Our purpose is to present the first examples.

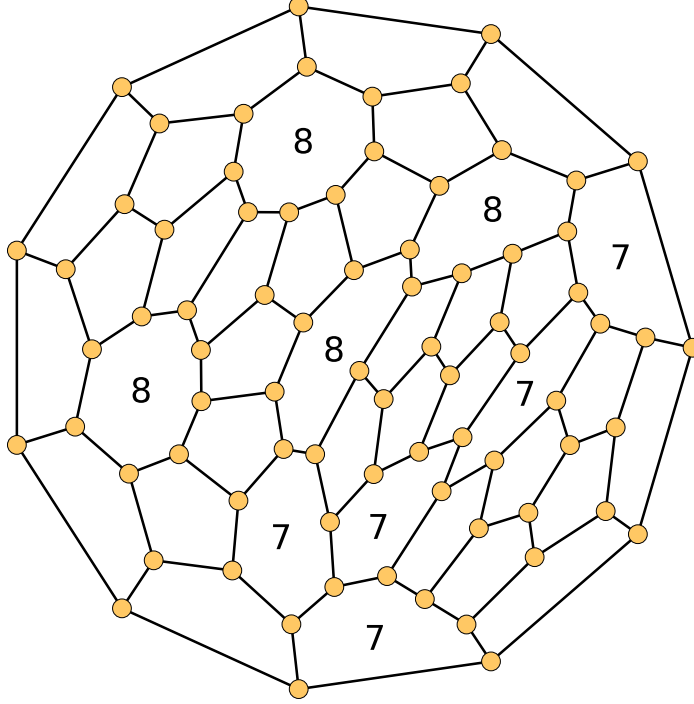


Figure 1: Hypohamiltonian graph with face count  $5^{30} 7^5 8^4 11$

## 2 The results

Using the program `plantri` [4, 5] we generated all planar cubic graphs of girth 5 and cyclic connectivity at least 4 on up to 76 vertices. No theory is known which could efficiently restrict the search to a small subclass sure to contain the hypohamiltonian graphs, so we just tested all of them.

The computational task was daunting as more than  $10^{13}$  graphs are involved and 76 vertices is large enough that a primitive backtrack search for hamiltonian cycles takes up to several hours per graph. Fortunately we have a program (unpublished) specifically designed for sub-cubic graphs that takes only 10 microseconds on average to find a hamiltonian cycle if there is one, and 15 milliseconds on average to rigorously prove that there is none. Even then the task took about 8 years of cpu time.

The results are shown in Table 1. For at most 74 vertices there are no hypohamiltonian graphs, but for 76 vertices there are three. All of them have cyclic connectivity 5. The graphs themselves are shown in Figures 1–3, which were drawn by CaGe [3].

Note that the graphs have different face counts despite having some similarities. The non-hamiltonicity of these graphs does not follow immediately from Grinberg’s condition, so we tested them with several independent programs to remove the possibility of error. Contrary to our usual experience of extremal graphs, none of them have any non-trivial automorphisms. We did not find a way to generalize these examples to larger sizes despite trying several approaches, so the problem of finding an infinite family remains open.

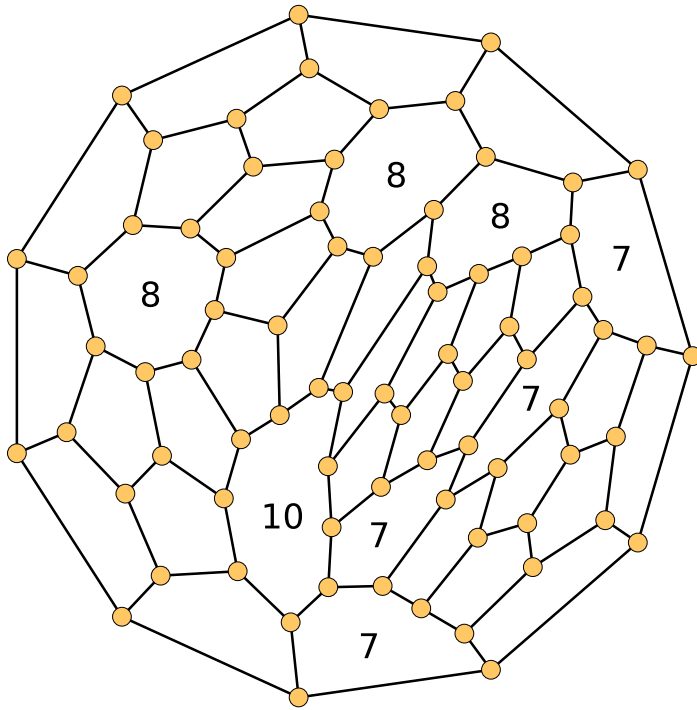


Figure 2: Hypohamiltonian graph with face count  $5^{31} 7^4 8^3 10 11$

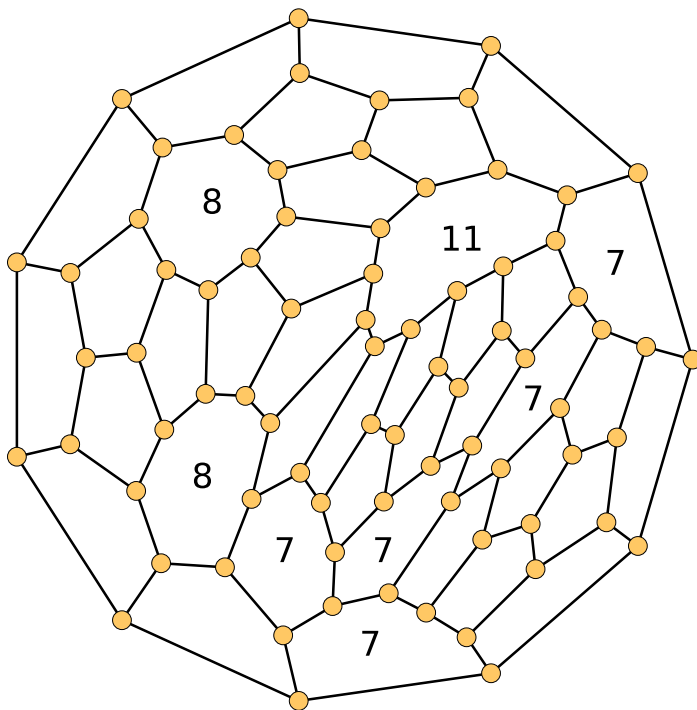


Figure 3: Hypohamiltonian graph with face count  $5^{31} 7^5 8^2 11^2$

$n$	$C_4(n)$	$N_4(n)$	$C_5(n)$	$N_5(n)$	$H(n)$
20	1		1		
22	0		0		
24	1		1		
26	1		1		
28	3		3		
30	4		4		
32	12		12		
34	23		23		
36	2		71		
38	4		187		
40	22		627		
42	84		1970		
44	376	1	6833	1	
46	1579	3	23384	1	
48	6751	1	82625	0	
50	27969	3	292164	3	
52	115423	6	1045329	6	
54	467948	12	3750277	2	
56	1882184	49	13532724	22	
58	7496828	126	48977625	37	
60	29667311	214	177919099	31	
62	116710547	659	648145255	194	
64	457122502	1467	2368046117	298	
66	1783850057	3247	8674199554	306	
68	6941579864	9187	31854078139	1538	
70	26950926431	22069	117252592450	2566	
72	104455609591	50514	432576302286	3091	
74	404298188921	137787	1599320144703	13487	
76	1563255455769	339804	5925181102878	22274	3

$n$  = the number of vertices

$C_4(n)$  = the number with cyclic connectivity exactly 4

$N_4(n)$  = the number of those which are not hamiltonian

$C_5(n)$  = the number with cyclic connectivity exactly 5

$N_5(n)$  = the number of those which are not hamiltonian

$H(n)$  = the number which are hypohamiltonian

The numbers  $C_4(n)$  and  $C_5(n)$  first appeared in [4].

Table 1: Counts of planar cyclically 4-connected cubic graphs of girth 5

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